Review

linear image transformation:

Let $g = \mathcal{O}(f)$

$$g(\alpha, \beta) = \sum_{x=1}^{N} \sum_{y=1}^{N} f(x, y)\, h(x, \alpha, y, \beta)$$

($g(\alpha,\beta)$ is linear combinations of all the entries of f for all α, β).

$h(\cdot, \alpha, \cdot, \beta)$ is called the PSF at (α, β).

Definition:

shift invariant $\Longleftrightarrow$

$h(x, \alpha, y, \beta) = \tilde{h}(\alpha - x, \beta - y)$ for any $1 \leq \alpha, \beta, x, y \leq N$.

Seperable $\Longleftrightarrow$

$h(x, \alpha, y, \beta) = h_c(x, \alpha)\, h_r(y, \beta)$

Q1:

Determine whether the followings is shift invariant or seperable?

1. $h(x, \alpha, y, \beta) = \alpha e^{\beta - y}$

2. $h(x, \alpha, y, \beta) = (\alpha - x) e^{(\alpha - x) - (\beta - y)}$

3. $h(x, \alpha, y, \beta) = \begin{cases} \sqrt{17 - (\alpha - x)^3 + (\beta - y)^2} & \text{if } \begin{array}{l} |\alpha - x| \leq 2 \\ |\beta - y| \leq 3 \end{array} \\ 0 & \text{otherwise} \end{cases}$

Answer:

1. Not shift invariant.

Take $\alpha = 2$, $x = 1$, $\beta = y = 1$,

$h(1, 2, 1, 1) = 2$

But, if $\alpha = 1$, $x = 0$, $\beta = y = 1$,

$h(0, 1, 1, 1) = 1$

Seperable:

$$h(x,\alpha,y,\beta) = \underbrace{(\alpha)}_{h_c}\underbrace{(e^{\beta-y})}_{h_r}$$

2. Shift invariant;

let $s = \alpha - x$, $t = \beta - y$,

we see $h(x,\alpha,y,\beta) = se^{s-t}$

Separable:

$$h(x,\alpha,y,\beta) = \underbrace{\left((\alpha - x)e^{\alpha-x}\right)}_{h_c}\underbrace{\left(e^{-(\beta-y)}\right)}_{h_r}$$

3. Shift-invariant:

Let $s = \alpha - x$, $t = \beta - y$,

$$h(x,\alpha,y,\beta) = \begin{cases} \sqrt{17 - s^2 + t^2} & \text{if } |s| \leq 2,\ |t| \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

Not separable:

We prove by contradiction:

suppose $h(x,\alpha,y,\beta) = h_c(x,\alpha)\, h_r(y,\beta)$

for some h_c, h_r.

Then, $h(1,1,2,2) = h_c(1,1)\, h_r(2,2) = \sqrt{17}$

$$h(1,1,2,1) = h_c(1,1)\, h_r(2,1) = 3\sqrt{2}$$

$$\Rightarrow \frac{h_r(2,2)}{h_r(2,1)} = \frac{\sqrt{17}}{3\sqrt{2}}$$

But, $h(2,1,2,2) = h_c(2,1)\, h_r(2,2) = 3\sqrt{2}$

$$h(2,1,2,1) = h_c(2,1)\, h_r(2,1) = \sqrt{19}$$

$$\Rightarrow \frac{h_r(2,2)}{h_r(2,1)} = \underbrace{\frac{3\sqrt{2}}{\sqrt{19}} \neq \frac{\sqrt{17}}{3\sqrt{2}}}_{\text{Contradiction}}.$$